

HOMOTOPY UNITS IN A -INFINITY ALGEBRAS

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ABSTRACT. We show that the space of unital associative DG-algebra structures on a DG-module is up to homotopy a subset of connected components of the space of non-unital associative DG-algebra structures. The analogue for DG-categories is also proven. For this we prove that the canonical map from the associative DG-operad to the unital associative DG-operad is a homotopy epimorphism.

INTRODUCTION

Strictly speaking, in order to define a unital associative algebra we must construct a binary product and specify a unit element. Nevertheless, we all know that an associative algebra cannot have more than one unit, therefore being unital is a property rather than a structure. That is to say, the set of unital associative algebra structures on a given module M is a subset the set of associative algebra structures,

$$\left\{ \begin{array}{c} \text{Unital associative algebra} \\ \text{structures on } M \end{array} \right\} \xrightarrow{\text{forget the unit}} \left\{ \begin{array}{c} \text{Associative algebra} \\ \text{structures on } M \end{array} \right\}.$$

These two sets carry an action of the automorphism group of the module M and isotropy groups correspond to (unital) associative algebra automorphisms.

In the differential graded (DG) context, the isomorphy relation is too coarse, and isomorphisms are replaced with quasi-isomorphisms. This has striking consequences, such as the existence of spaces, rather than sets, of (unital) associative DG-algebra structures on a given a DG-module M . The main result of this paper shows that the map

$$\left\{ \begin{array}{c} \text{Unital associative DG-algebra} \\ \text{structures on } M \end{array} \right\} \xrightarrow{\text{forget the unit}} \left\{ \begin{array}{c} \text{Associative DG-algebra} \\ \text{structures on } M \end{array} \right\}$$

is a homotopy equivalence from the source to a certain subset of connected components of the target (Corollary 2.4). This is the strongest generalization of the ungraded case one can expect.

We work over an arbitrary commutative ring \mathbb{k} . Different kinds of algebras are better compared by using the language of operads. Each operad \mathcal{O} encodes a kind of algebra structure, e.g. we have an operad **Ass** for associative algebras and another

1991 *Mathematics Subject Classification.* 18D50, 18G55.

Key words and phrases. Differential graded algebra, A -infinity algebra, differential graded category, A -infinity category, operad, algebra, model category, mapping space.

The author was partially supported by the Andalusian Ministry of Economy, Innovation and Science under the grant FQM-5713, by the Spanish Ministry of Education and Science under the MEC-FEDER grant MTM2010-15831, and by the Government of Catalonia under the grant SGR-119-2009.

operad \mathbf{uAss} for unital associative algebras. An \mathbf{O} -algebra structure on M is the same as an operad morphism $\mathbf{O} \rightarrow \mathbf{End}_{\mathbb{k}}(M)$, where $\mathbf{End}_{\mathbb{k}}(M)$ is the endomorphism operad of M ; in the same way that a module structure over an associative algebra is a morphism to an endomorphism algebra.

DG-operads have a homotopy theory, i.e. the category $\mathbf{Op}(\mathbb{k})$ of DG-operads over \mathbb{k} has a combinatorial model category structure [Mur11]. The space of \mathbf{O} -algebra structures on a DG-module M is the derived mapping space $\mathrm{Map}_{\mathbf{Op}(\mathbb{k})}(\mathbf{O}, \mathbf{End}_{\mathbb{k}}(M))$ in the sense of [DK80]. The comparison between the spaces of unital and non-unital associative DG-algebra structures is made through the operad morphism $\psi: \mathbf{Ass} \rightarrow \mathbf{uAss}$ which models that we can forget the unit, see Example 1.4.

The main result follows from an intrinsic property of the morphism $\psi: \mathbf{Ass} \rightarrow \mathbf{uAss}$. It is a homotopy epimorphism (Theorem 2.3). This general approach makes possible to extend our main result to other contexts, such as DG-categories (Corollary 2.7).

For the computation of the space of \mathbf{O} -algebra structures we need a cofibrant resolution of \mathbf{O} in $\mathbf{Op}(\mathbb{k})$. The operad \mathbf{A}_{∞} for Stasheff's A -infinity algebras is a cofibrant resolution of \mathbf{Ass} . Moreover, a cofibrant resolution of \mathbf{uAss} is given by the operad \mathbf{uA}_{∞} for Fukaya–Oh–Ohta–Ono homotopy unital A -infinity algebras, see [Lyu11, MT11]. Furthermore, in order to show that $\psi: \mathbf{Ass} \rightarrow \mathbf{uAss}$ is a homotopy epimorphism we really need to work with a cofibrant replacement $\tilde{\psi}: \mathbf{A}_{\infty} \rightarrow \mathbf{uA}_{\infty}$. This is the reason for the title of this paper.

1. OPERADS

This section contains some basic notions on differential graded operads. Notice that we work with non-symmetric operads.

Definition 1.1. A *graded operad* \mathbf{O} is a sequence of graded modules $\mathbf{O}(n)$, $n \geq 0$, together with composition laws,

$$\circ_i: \mathbf{O}(p) \otimes_{\mathbb{k}} \mathbf{O}(q) \longrightarrow \mathbf{O}(p+q-1), \quad 1 \leq i \leq p, \quad q \geq 0,$$

and a unit element $\mathrm{id}_{\mathbf{O}} \in \mathbf{O}(1)_0$ satisfying the following relations:

- (1) $(a \circ_i b) \circ_j c = (-1)^{|b||c|} (a \circ_j c) \circ_{i+q-1} b$ if $1 \leq j < i$ and $c \in \mathbf{O}(q)$.
- (2) $(a \circ_i b) \circ_j c = a \circ_i (b \circ_{j-i+1} c)$ if $b \in \mathbf{O}(p)$ and $i \leq j < p+i$.
- (3) $\mathrm{id}_{\mathbf{O}} \circ_1 a = a$.
- (4) $a \circ_i \mathrm{id}_{\mathbf{O}} = a$.

If there is no risk of confusion we simply denote the unit element by id .

A graded operad *morphism* $f: \mathbf{O} \rightarrow \mathbf{P}$ is a sequence of degree 0 homomorphisms $f(n): \mathbf{O}(n) \rightarrow \mathbf{P}(n)$ such that

$$f(1)(\mathrm{id}_{\mathbf{O}}) = \mathrm{id}_{\mathbf{P}},$$

and if $a \in \mathbf{O}(p)$ and $b \in \mathbf{O}(q)$ then for $1 \leq i \leq p$,

$$f(p+q-1)(a \circ_i b) = f(p)(a) \circ_i f(q)(b).$$

Given two graded operad morphisms $f, g: \mathbf{O} \rightarrow \mathbf{P}$, an (f, g) -*derivation* $d: \mathbf{O} \rightarrow \mathbf{P}$ of degree ± 1 is a sequence of degree ± 1 homomorphisms $d(n): \mathbf{O}(n) \rightarrow \mathbf{P}(n)$ such that if $a \in \mathbf{O}(p)$ and $b \in \mathbf{O}(q)$ then for $1 \leq i \leq p$,

$$d(p+q-1)(a \circ_i b) = d(p)(a) \circ_i g(q)(b) + (-1)^{|a|} f(p)(a) \circ_i d(q)(b).$$

Notice that $d(1)(\mathrm{id}_{\mathbf{O}}) = 0$.

A *differential graded operad*, a.k.a. DG-operad, is a graded operad together with a $(1_0, 1_0)$ -derivation $d_0: \mathbf{0} \rightarrow \mathbf{0}$ of degree -1 with $d_0^2 = 0$. Again, if there is no risk of confusion we denote this derivation just by d .

A *morphism* of DG-operads $f: \mathbf{0} \rightarrow \mathbf{P}$ is a morphism between the underlying graded operads commuting with the differentials, $fd_0 = d_P f$.

A *homotopy* $h: f \Rightarrow g$ between two DG-operad morphisms $f, g: \mathbf{0} \rightarrow \mathbf{P}$ is an (f, g) -derivation $h: \mathbf{0} \rightarrow \mathbf{P}$ of degree $+1$ of the underlying graded operads such that $f - g = d_P h + h d_0$.

A *strong deformation retraction* of DG-operads is a diagram

$$\mathbf{0} \begin{array}{c} \xrightarrow{l} \\ \xleftarrow{r} \end{array} \mathbf{P} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} h$$

consisting of two morphisms l and r , and a homotopy $h: lr \Rightarrow 1_P$ such that $h(n)l(n) = 0$, $n \geq 0$, in particular $rl = 1_0$.

Example 1.2. The *endomorphism operad* $\mathbf{End}_k(M)$ of a module M is given by

$$\mathbf{End}_k(M)(n) = \mathrm{Hom}_k(M \otimes_k \cdots \otimes_k M, M).$$

The composition law

$$\circ_i: \mathbf{End}_k(M)(p) \otimes_k \mathbf{End}_k(M)(q) \longrightarrow \mathbf{End}_k(M)(p+q-1), \quad 1 \leq i \leq p,$$

is defined by

$$a \circ_i b = a(1_M^{\otimes(i-1)} \otimes_k b \otimes_k 1_M^{\otimes(p-i)}).$$

If M is a DG-module then $\mathbf{End}_k(M)$ is a DG-operad, using the internal tensor product \otimes_k and Hom_k in the category of DG-modules.

Example 1.3. The operad \mathbf{uAss} whose algebras are unital associative k -algebras is given by $\mathbf{uAss}(n) = k$ for all $n \geq 0$. All composition laws $\circ_i: k \otimes_k k \rightarrow k$ are defined by the product in k . This operad can be regarded as a DG-operad concentrated in degree 0.

If M is a \mathbf{uAss} -algebra defined by an operad morphism $f: \mathbf{Ass} \rightarrow \mathbf{End}_k(M)$, the associative product $\mu: M \otimes_k M \rightarrow M$ is defined by

$$\begin{aligned} f(2): k &\longrightarrow \mathrm{Hom}_k(M \otimes_k M, M), \\ 1 &\mapsto \mu. \end{aligned}$$

The unit for μ is defined by

$$f(0): k \longrightarrow \mathrm{Hom}_k(k, M).$$

It is $f(0)(1)(1) \in M$.

Example 1.4. The operad \mathbf{Ass} whose algebras are associative k -algebras is given by $\mathbf{Ass}(0) = 0$ and $\mathbf{Ass}(n) = k$ if $n > 0$. Composition laws not involving $\mathbf{Ass}(0)$ are defined as above, so that there is an operad morphism

$$\psi: \mathbf{Ass} \longrightarrow \mathbf{uAss}$$

defined by $\psi(n) = 1_k$ for $n > 0$.

If M is an \mathbf{Ass} -algebra defined by an operad morphism $f: \mathbf{Ass} \rightarrow \mathbf{End}_k(M)$, the associative product $\mu: M \otimes_k M \rightarrow M$ is defined as in the previous example. Therefore, the pull-back of algebra structure along the operad morphism ψ consists of forgetting the unit in a unital associative k -algebra.

We regard the operad \mathbf{Ass} as a DG-operad concentrated in degree 0, and ψ as a morphism of DG-operads.

The model structure in the category $\mathbf{Op}(\mathbb{k})$ of DG-operads is determined by the following data: a morphism $f: \mathbf{0} \rightarrow \mathbf{P}$ is a weak equivalence (resp. fibration) if $f(n)$ is a quasi-isomorphism (resp. levelwise surjection) for all $n \geq 0$. In particular, the morphisms r and l in a strong deformation retraction are weak equivalences. Moreover, the homotopies defined above are the right homotopies in the model category $\mathbf{Op}(\mathbb{k})$.

This model structure in $\mathbf{Op}(\mathbb{k})$ is transferred from the model category $\mathbf{Ch}(\mathbb{k})$ of DG-modules, which is combinatorial. Sets of generating cofibrations I and acyclic cofibrations J of $\mathbf{Ch}(\mathbb{k})$ are

$$I = \left\{ \begin{array}{c} \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \\ \downarrow \quad \downarrow \quad \downarrow \\ \cdots \rightarrow 0 \rightarrow \mathbb{k} \rightarrow 0 \rightarrow \cdots \\ \text{degree } n \end{array} ; n \in \mathbb{Z} \right\},$$

$$J = \left\{ \begin{array}{c} \cdots \rightarrow 0 \rightarrow \mathbb{k} \rightarrow 0 \rightarrow \cdots \\ \downarrow \quad \downarrow_{1_{\mathbb{k}}} \quad \downarrow \\ \cdots \rightarrow \mathbb{k} \xrightarrow{1_{\mathbb{k}}} \mathbb{k} \rightarrow 0 \rightarrow \cdots \\ \text{degree } n \end{array} ; n \in \mathbb{Z} \right\}.$$

Therefore, $\mathbf{Op}(\mathbb{k})$ is also combinatorial [Mur11]. Moreover, cofibrant objects are retracts of DG-operads with free underlying graded operad, which will be considered in detail in Section 3. Furthermore, cofibrations are retracts of DG-operad morphisms $f: \mathbf{0} \rightarrow \mathbf{P}$ such that the graded operad underlying \mathbf{P} is constructed from $\mathbf{0}$ by adding new generators and no relations, and f is the inclusion.

2. HOMOTOPY EPIMORPHISMS

In this section we use some elementary homotopy theory in model categories. Two standard references are [Hov99] and [Hir03].

The following definition of homotopy epimorphism is dual of the notion of homotopy monomorphism in [Toë07].

Definition 2.1. A morphism $f: X \rightarrow Y$ in a model category \mathcal{M} is said to be a *homotopy epimorphism* if for any object Z in \mathcal{M} , the induced morphism on derived mapping spaces,

$$f^* = \mathrm{Map}_{\mathcal{M}}(f, Z): \mathrm{Map}_{\mathcal{M}}(Y, Z) \longrightarrow \mathrm{Map}_{\mathcal{M}}(X, Z),$$

induces an injection on connected components,

$$\pi_0 f^*: \pi_0 \mathrm{Map}_{\mathcal{M}}(Y, Z) \hookrightarrow \pi_0 \mathrm{Map}_{\mathcal{M}}(X, Z),$$

and isomorphisms on homotopy groups for all possible base points $x \in \mathrm{Map}_{\mathcal{M}}(Y, Z)$,

$$\pi_n f^*: \pi_n(\mathrm{Map}_{\mathcal{M}}(Y, Z), x) \xrightarrow{\cong} \pi_n(\mathrm{Map}_{\mathcal{M}}(X, Z), f^*x), \quad n \geq 1.$$

These conditions on homotopy groups/sets are equivalent to say that f^* corestricts to a weak equivalence from the source to a subset of connected components of the target.

The construction of derived mapping spaces we have in mind is the simplicial set

$$\mathrm{Map}_{\mathcal{M}}(X, Z) = \mathcal{M}(\tilde{X}, Y_{\bullet}),$$

where \tilde{X} is a cofibrant resolution of X and Y_\bullet is a simplicial resolution of Y [DK80].

The following lemma is an intrinsic characterization of homotopy epimorphisms. It is equivalent to the dual characterization of homotopy monomorphisms in terms of the derived diagonal noticed in [Toë07].

Lemma 2.2. *A morphism $f: X \rightarrow Y$ in a model category \mathcal{M} is a homotopy epimorphism if and only if the inclusion of the first factor $\phi_1^h: Y \rightarrow Y \cup_X^h Y$ of the derived coproduct over X is an isomorphism in the homotopy category $\text{Ho } \mathcal{M}$.*

Recall that the honest inclusions of the factors $\phi_1, \phi_2: Y \rightarrow Y \cup_X Y$ are defined by the following push-out diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & \text{push} & \downarrow \phi_2 \\ Y & \xrightarrow{\phi_1} & Y \cup_X Y. \end{array}$$

The inclusions in the derived coproduct $\phi_1^h: Y \rightarrow Y \cup_X^h Y$ are defined by the honest inclusions of a cofibrant replacement \tilde{f} of f , which is given by a commutative square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \sim \downarrow c_X & & c_Y \downarrow \sim \\ X & \xrightarrow{f} & Y \end{array}$$

where the vertical maps are trivial fibrations and the upper horizontal arrow is a cofibration between cofibrant objects. More precisely, $\phi_1^h = \phi_1 c_Y^{-1}$ in $\text{Ho } \mathcal{M}$.

Proof of Lemma 2.2. Consider the following push-out square in \mathcal{M} ,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ \tilde{f} \downarrow & \text{push} & \downarrow \phi_2 \\ \tilde{Y} & \xrightarrow{\phi_1} & \tilde{Y} \cup_{\tilde{X}} \tilde{Y} = Y \cup_X^h Y. \end{array}$$

If we apply $\text{Map}_{\mathcal{M}}(-, Z)$ we obtain a pull-back of simplicial sets formed by Kan fibrations,

$$\begin{array}{ccc} \text{Map}_{\mathcal{M}}(X, Z) & \xleftarrow{f^*} & \text{Map}_{\mathcal{M}}(Y, Z) \\ f^* \uparrow & \text{pull} & \uparrow \phi_2^* \\ \text{Map}_{\mathcal{M}}(Y, Z) & \xleftarrow{\phi_1^*} & \text{Map}_{\mathcal{M}}(Y \cup_X^h Y, Z). \end{array}$$

In a such a pull-back, parallel arrows have essentially the same homotopy fibers. More precisely, if F_x denotes the homotopy fiber of ϕ_1^* over a base point $x \in \text{Map}_{\mathcal{M}}(Y, Z)_0$, then F_x is also the homotopy fiber of f^* over f^*x . Hence, by the long exact sequence in homotopy groups, f is a homotopy epimorphism if and only if F_x is weakly contractible for any x .

The map $\pi_0 \phi_1^*$ is surjective since ϕ_1 splits the codiagonal

$$\nabla = (1_{\tilde{Y}}, 1_{\tilde{Y}}): \tilde{Y} \cup_{\tilde{X}} \tilde{Y} \rightarrow \tilde{Y},$$

defined by the universal property of the previous push-out, i.e. $\nabla\phi_1 = 1_{\tilde{Y}}$, so $(\pi_0\phi_1^*)(\pi_0\nabla^*)$ is the identity. Therefore, F_x is weakly contractible for any x if and only if ϕ_1^* is a weak equivalence of spaces. This is true for all Z if and only if ϕ_1 is a weak equivalence in \mathcal{M} , which is the same as ϕ_1^h being an isomorphism in $\text{Ho}\mathcal{M}$. \square

Theorem 2.3. *The DG-operad morphism $\psi: \mathbf{Ass} \rightarrow \mathbf{uAss}$ in Example 1.4 is a homotopy epimorphism.*

Proof. We use the characterization of homotopy epimorphisms in Lemma 2.2. In Section 4 we recall from [MT11] a cofibrant replacement $\tilde{\psi}$ of ψ and decompose the inclusion of the first factor ϕ_1 as a transfinite composition of trivial cofibrations. Actually, each of these trivial cofibrations is the inclusion of a strong deformation retraction, see Lemma 4.1. Therefore ϕ_1 is a weak equivalence, and hence ϕ_1^h is an isomorphism in the homotopy category of DG-operads $\text{Ho Op}(\mathbb{k})$. \square

If we use the condition defining homotopy epimorphisms (Definition 2.1) with an endomorphism operad in the target we obtain the main result of this paper.

Corollary 2.4. *For any DG-module M , the map*

$$\psi^*: \text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{uAss}, \mathbf{End}_{\mathbb{k}}(M)) \longrightarrow \text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{Ass}, \mathbf{End}_{\mathbb{k}}(M)),$$

induces a weak equivalence from the source to a subset of connected components of the target.

Remark 2.5. All DG-operads $\mathbf{0}$ are fibrant in $\text{Op}(\mathbb{k})$, and we can easily construct simplicial resolutions as follows. Standard simplices $\Delta^0, \Delta^1, \Delta^2, \dots$ form a cosimplicial space Δ^\bullet , i.e. a cosimplicial simplicial set. Denote $C^*(-, \mathbb{k})$ the normalized cochains functor from simplicial sets to (unital associative) DG-algebras, and consider the simplicial DG-algebra $C^*(\Delta^\bullet, \mathbb{k})$. One can construct new DG-operads from $\mathbf{0}$ by extension of scalars. We can extend scalars from \mathbb{k} to any \mathbb{k} -algebra, or even DG-algebra. The simplicial DG-operad $\mathbf{0} \otimes_{\mathbb{k}} C^*(\Delta^\bullet, \mathbb{k})$ is a simplicial resolution of $\mathbf{0}$, and the operad of vertices is $\mathbf{0} \otimes_{\mathbb{k}} C^*(\Delta^0, \mathbb{k}) = \mathbf{0}$ itself.

Remark 2.6. A connected component of $\text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{Ass}, \mathbf{End}_{\mathbb{k}}(M))$ may not be represented by an honest associative DG-algebra structure on M , but it is always represented by an A -infinity algebra structure, since the DG-operad \mathbf{A}_∞ for Stasheff's A -infinity algebras is a cofibrant resolution of \mathbf{Ass} , compare Section 4. The same holds in the unital case, using the notion of homotopy unital A -infinity algebras of [FOOO09a, FOOO09b], see [Lyu11, MT11] and again Section 4.

If we take the simplicial resolution of $\mathbf{End}_{\mathbb{k}}(M)$ in Remark 2.5, then a 1-simplex in $\text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{Ass}, \mathbf{End}_{\mathbb{k}}(M))$ between two A -infinity algebra structures on M is the same as an A -infinity morphism whose linear part is the identity in M . Therefore, if \mathbb{k} is a field, Kadeishvili's theorem [Kad80] and [LH03, Théorème 3.2.1.1] show that the connected components in the image of ψ^* are those corresponding to A -infinity algebra structures on M inducing a unital graded algebra structure on homology H_*M .

We can also consider DG-modules with several objects [Mur11, §10]. A DG-module M with object set S is a collection of DG-modules indexed by $S \times S$,

$$M = \{M(x, y)\}_{x, y \in S}.$$

A morphism $f: M \rightarrow N$ in the category $\text{Graph}_S(\mathbb{k})$ of DG-modules with object set S is just a collection of morphisms $f(x, y): M(x, y) \rightarrow N(x, y)$, $x, y \in S$. This category has a tensor product $\otimes_{\mathbb{k}[S]}$ defined by

$$(M \otimes_{\mathbb{k}[S]} N)(x, y) = \bigoplus_{z \in S} M(z, y) \otimes_{\mathbb{k}} N(x, z).$$

The morphism (honest) DG-module of two DG-modules with object set S is

$$\text{Hom}_{\mathbb{k}[S]}(M, N) = \prod_{x, y \in S} \text{Hom}_{\mathbb{k}}(M(x, y), N(x, y)),$$

and the endomorphism DG-operad $\text{End}_{\mathbb{k}[S]}(M)$ of a DG-module with object set S is defined by using $\otimes_{\mathbb{k}[S]}$ and $\text{Hom}_{\mathbb{k}[S]}$ as in Example 1.2.

An **uAss**-algebra structure on a DG-module M with object set S is a DG-category with objects S and morphism DG-modules $M(x, y)$, $x, y \in S$. Similarly, an **Ass**-algebra structure on M is a DG-category without identity morphisms.

The following corollary of Theorem 2.3 is a categorical version of our main result.

Corollary 2.7. *For any DG-module M with object set S , the map*

$$\psi^*: \text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{uAss}, \mathbf{End}_{\mathbb{k}[S]}(M)) \longrightarrow \text{Map}_{\text{Op}(\mathbb{k})}(\mathbf{Ass}, \mathbf{End}_{\mathbb{k}[S]}(M)),$$

induces a weak equivalence from the source to a subset of connected components of the target.

The obvious categorical version of Remark 2.6 applies to this corollary.

3. FREE OPERADS

DG-operads with free underlying graded operad are the paradigmatic cofibrant objects in $\text{Op}(\mathbb{k})$. Cofibrant resolutions of operads are a main tool in this paper. This section contains most of what the reader needs to know.

The free graded module on a graded set $S = \coprod_{j \in \mathbb{Z}} S_j$ is

$$\mathbb{k} \cdot S = \bigoplus_{j \in \mathbb{Z}} \mathbb{k} \cdot S_j.$$

A graded operad $\mathbf{0}$ is *free* with basis $B = \coprod_{n \geq 0} B(n)$ if $B(n)$ is a graded set, $B(n)_j \subset \mathbf{0}(n)_j$, and given a graded operad \mathbf{P} , any collection of maps

$$f(n)_j: B(n)_j \longrightarrow \mathbf{P}(n)_j$$

can be uniquely extended to a graded operad morphism $f: \mathbf{0} \rightarrow \mathbf{P}$. Free graded operads can be explicitly described as follows, compare [Mur11].

A *planted tree with leaves* is a contractible finite 1-dimensional simplicial complex T with a set of vertices $V(T)$, a non-empty set of edges $E(T)$, a distinguished vertex $r(T) \in V(T)$ called *root*, and a set of distinguished vertices $L(T) \subset V(T) \setminus \{r(T)\}$ called *leaves*. The root and the leaves must have degree 1. The other degree one vertices are called *corks*. Recall that the *degree* of $v \in V(T)$ is the number of edges containing v , and

$$\tilde{v} = (\text{degree of } v) - 1.$$

An *inner vertex* is a vertex which is neither a leaf nor the root, and an inner edge is an edge whose vertices are inner. The set of inner vertices will be denoted by $I(T)$,

$$V(T) = \{r(T)\} \coprod I(T) \coprod L(T).$$

The *level* of a vertex $v \in V(T)$ is the distance to the root, $\text{level}(v) = d(v, r(T))$, with respect to the usual metric d such that the distance between two adjacent vertices $\{u, v\} \in E(T)$ is $d(u, v) = 1$.

A *planted planar tree* with leaves is a planted tree with leaves together with a total order \preceq on $V(T)$, called *path order*, satisfying the following conditions. Given two vertices $v, w \in V(T)$:

- If v lies in the shortest path between $r(T)$ and w then $v \prec w$.
- Otherwise, assume that the shortest path from $r(T)$ to v coincides with the shortest path from $r(T)$ to w up to level $n \geq 1$. Let v' (resp. w') be the level $n + 1$ vertex in the shortest path from $r(T)$ to v (resp. w). If $v' \prec w'$ then $v \prec w$.

An *isomorphism* of planted planar trees with leaves is a simplicial isomorphism preserving the path order, the leaves, and the root.

All trees in this paper will be planted planar trees with leaves.

The geometric realization $|T|$ of a tree is depicted by drawing vertices with the same level on the same horizontal line, following from left to right the order induced by the path order (see Figure 1). For most purposes, the heuristic picture we should have in mind of a tree T corresponds to the space $\|T\| = |T| \setminus (\{r(T)\} \sqcup L(T))$.

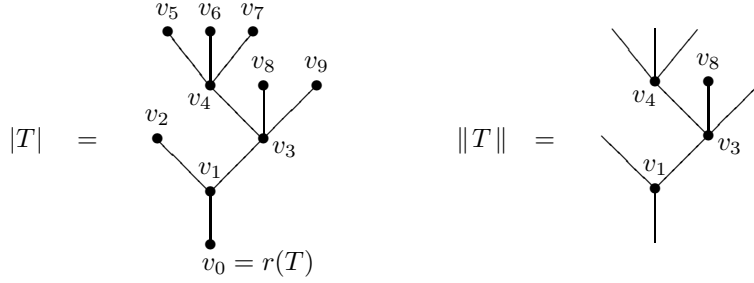


FIGURE 1. On the left, the geometric realization of a tree T with vertices ordered by the subscript. The set of leaves is $L(T) = \{v_2, v_5, v_6, v_7, v_9\}$. On the right, the space $\|T\|$, where we can see the only cork v_8 .

Given trees T and T' with p and q leaves, respectively, and $1 \leq i \leq p$, the tree $T \circ_i T'$ is obtained by *grafting* the edge of $\|T'\|$ containing the root onto the edge of $\|T\|$ containing the i^{th} leaf. The path order in $V(T \circ_i T')$ is obtained by inserting $V(T') \setminus \{r(T')\}$ into $V(T)$ in place of the i^{th} leaf. The unit for the grafting operation is tree T with $\|T\| = |$. Notice that $I(T \circ_i T') = I(T) \amalg I(T')$.

The free graded operad with basis B is given by

$$\mathcal{O}(n) = \bigoplus_T \bigotimes_{v \in I(T)} \mathbb{k} \cdot B(\tilde{v}).$$

Here T runs over the set of isomorphism classes of trees with n leaves. The composition law

$$\circ_i : \mathcal{O}(p) \otimes_{\mathbb{k}} \mathcal{O}(q) \longrightarrow \mathcal{O}(p + q - 1)$$

sends the tensor product of the factors corresponding to the trees T and T' isomorphically to the factor of $T \circ_i T'$. The unit $\text{id} \in \mathcal{O}(1)_0$ is the generator of the direct summand of $\|T\| = |$.

A free graded operad is filtered according to the number of inner vertices of trees. This is called the *weight* filtration. The elements of weight 0 are $\mathbb{k} \cdot \text{id}$, and $B(n)_j \subset \mathcal{O}(n)_j$ are the generators of weight 1.

Lemma 3.1. *Let \mathcal{O} be a DG-operad whose underlying graded operad is free with basis B such that $B(n)_j = \emptyset$ for $j < 0 \leq n$. Given a DG-operad morphism $g: \mathcal{O} \rightarrow \mathcal{P}$ and maps*

$$(3.2) \quad h(n)_j: B(n)_j \longrightarrow \mathcal{P}(n)_{j+1}, \quad n, j \geq 0,$$

there exists a unique DG-operad morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ and homotopy $h: f \Rightarrow g$ extending (3.2).

Proof. We define $h(n)_j$ and $f(n)_j$ on weight-homogeneous elements by induction on (j, weight) with respect to the lexicographic order.

On weight 0 elements, the homotopy h must be trivial and f is defined by the formula $f(1)_0(\text{id}_0) = \text{id}_{\mathcal{P}}$.

The homotopy is defined on weight 1 elements by (3.2). The homomorphism $f(n)_0$ is defined on weight 1 elements by the formula $f(n)_0 = g(n)_0 + d_{\mathcal{P}}(n)_1 h(n)_0$, since the equation $f = g + d_{\mathcal{P}}h + h d_0$ must hold and $d(n)_0 = 0$.

Assume we have defined $h(n)_0$ and $f(n)_0$ on elements of weight $< m$. Any weight m element in $\mathcal{O}(n)_0$ is a sum of elements of the form $a \circ_i b$ with $a \in \mathcal{O}(p)_0$ of weight $m-1$ and $b \in \mathcal{O}(q)_0$ of weight 1, $p+q = n+1$, $1 \leq i \leq p$. Therefore we must set

$$h(n)_0(a \circ_i b) = h(p)_0(a) \circ_i g(q)_0(b) + f(p)_0(a) \circ_i h(q)_0(b)$$

so that the derivation equation holds. Again, we must define $f(n)_0$ on weight m elements by the formula $f(n)_0 = g(n)_0 + d_{\mathcal{P}}(n)_1 h(n)_0$.

Suppose that we have defined $h(n)_0, f(n)_0, \dots, h(n)_{j-1}, f(n)_{j-1}$. As we have already said, we must define $h(n)_j$ on weight 1 elements by (3.2). The homomorphism $f(n)_j$ is defined on weight 1 elements by the formula $f(n)_j = g(n)_j + d_{\mathcal{P}}(n)_{j+1} h(n)_j + h(n)_{j-1} d_0(n)_j$, since the equation $f = g + d_{\mathcal{P}}h + h d_0$ must hold.

Assume we have defined $h(n)_j$ and $f(n)_j$ on elements of weight $< m$. Any weight m element in $\mathcal{O}(n)_j$ is a sum of elements of the form $a \circ_i b$ with $a \in \mathcal{O}(p)_s$ of weight $m-1$ and $b \in \mathcal{O}(q)_t$ of weight 1, $p+q = n+1$, $1 \leq i \leq p$, $s+t = j$, $s, t \geq 0$. Therefore we must set

$$h(n)_{s+t}(a \circ_i b) = h(p)_s(a) \circ_i g(q)_t(b) + (-1)^s f(p)_s(a) \circ_i h(q)_t(b)$$

so that the derivation equation holds. Again, we must define $f(n)_j$ on weight m elements by the formula $f(n)_j = g(n)_j + d_{\mathcal{P}}(n)_{j+1} h(n)_j + h(n)_{j-1} d_0(n)_j$. This completes the inductive definition of $h(n)_j$ and $f(n)_j$.

A tedious but straightforward computation shows that f and h so defined are a DG-operad morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ and a homotopy $h: f \Rightarrow g$. Uniqueness has been checked during the definition. \square

4. THE TECHNICAL LEMMA

Let \mathbf{A}_{∞} be the DG-operad freely generated as a graded operad by

$$\mu_n \in \mathbf{A}_{\infty}(n)_{n-2}, \quad n \geq 2.$$

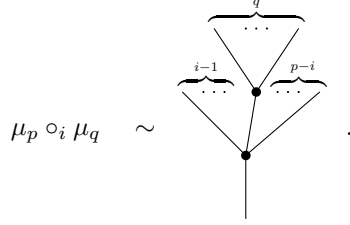
The differential is defined as

$$d(\mu_n) = \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} (-1)^{qp+(q-1)i} \mu_p \circ_i \mu_q.$$

If we identify μ_n with the corolla with n leaves,



the terms of $d(\mu_n)$ are all graftings of two corollas such that when we contract the inner edge we obtain μ_n ,



Now, let \mathbf{uA}_∞ be the DG-operad whose underlying graded operad is freely generated by

$$\mu_n^S \in \mathbf{A}_\infty(n - |S|)_{n-2+|S|}, \quad n > 0, \quad S \subset \{1, \dots, n\}, \quad (n, S) \neq (1, \emptyset).$$

The differential is given by

$$d(\mu_2^{\{j\}}) = \mu_2^\emptyset \circ_j \mu_1^{\{1\}} - \text{id}, \quad j \in \{1, 2\},$$

and if $(n, |S|) \neq (2, 1)$,

$$d(\mu_n^S) = \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p-|S_1| \\ S_1 \circ_i S_2 = S}} (-1)^{q(p-|S_1|)+(q-1)(i+r-1)+|S_2|(r-1)} \mu_p^{S_1} \circ_i \mu_q^{S_2}.$$

Here, if

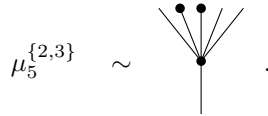
$$S_1 = \{j_1, \dots, j_s\} \subset \{1, \dots, p\}, \quad S_2 = \{k_1, \dots, k_t\} \subset \{1, \dots, q\},$$

and the i^{th} element of the complement of S_1 lies between j_{r-1} and j_r , then $S_1 \circ_i S_2 \subset \{1, \dots, n\}$ is

$$S_1 \circ_i S_2 = \{j_1, \dots, j_{r-1}, k_1 + i + r - 2, \dots, k_t + i + r - 2, j_r + q - 1, \dots, j_s + q - 1\}.$$

We understand that $r = 1$ if $i < j_1$ or $S_1 = \emptyset$, and $r = s + 1$ if the i^{th} element of the complement of S_1 is bigger than j_s .

We identify μ_n^S with the tree obtained from the corolla with n leaves by adding corks in the places indicated by S . The resulting tree has $n - |S|$ leaves,



For $n = 1$ the cork is drawn in white to avoid confusion in what follows,

$$\mu_1^{\{1\}} \sim \begin{array}{c} \circ \\ | \end{array}.$$

For $(n, |S|) \neq (2, 1)$, the terms of $d(\mu_n^S)$ are of two kinds. On the one hand, we have all graftings of two corollas with corks such that when we contract the unique inner edge which does not contain any cork we obtain μ_n^S ,

$$\mu_4^{\{2\}} \circ_2 \mu_2^{\{1\}} \sim \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array}.$$

On the other hand, we have the corolla with corks μ_n^S with a single cork turned white in all possible ways,

$$\mu_5^{\{2\}} \circ_2 \mu_1^{\{1\}} \sim \begin{array}{c} \bullet \quad \circ \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array}.$$

For $(n, |S|) = (2, 1)$,

$$d \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} \right) = \begin{array}{c} \circ \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} - \begin{array}{c} | \end{array}, \quad \left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} \right) = \begin{array}{c} \bullet \quad \circ \\ \diagdown \quad \diagup \\ \bullet \\ | \end{array} - \begin{array}{c} | \end{array}.$$

Here $| = \text{id}$ denotes the identity of the operad, which does not appear in $d(\mu_n^S)$ for $(n, |S|) \neq (2, 1)$.

The operad morphism

$$\begin{aligned} \tilde{\psi}: \mathbf{A}_\infty &\longrightarrow \mathbf{uA}_\infty, \\ \mu_n &\mapsto \mu_n^\emptyset, \end{aligned}$$

is a cofibrant replacement of $\psi: \mathbf{Ass} \rightarrow \mathbf{uAss}$, compare [MT11],

$$\begin{array}{ccc} \mathbf{A}_\infty & \xrightarrow{\tilde{\psi}} & \mathbf{uA}_\infty \\ \sim \downarrow & & \downarrow \sim \\ \mathbf{Ass} & \xrightarrow{\psi} & \mathbf{uAss}. \end{array}$$

The vertical trivial fibrations are

$$\begin{array}{ll} \mathbf{A}_\infty \longrightarrow \mathbf{Ass}, & \mathbf{uA}_\infty \longrightarrow \mathbf{uAss}, \\ \mu_2 \mapsto 1 \in \mathbf{Ass}(2) = \mathbb{k}, & \mu_2^\emptyset \mapsto 1 \in \mathbf{uAss}(2) = \mathbb{k}, \\ \mu_n \mapsto 0, \quad \text{otherwise}, & \mu_1^{\{1\}} \mapsto 1 \in \mathbf{uAss}(0) = \mathbb{k}, \\ & \mu_n^S \mapsto 0, \quad \text{otherwise}. \end{array}$$

Therefore,

$$\mathbf{uAss} \cup_{\mathbf{Ass}}^h \mathbf{uAss} = \mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty.$$

The image of generators by the inclusions

$$\phi_i: \mathbf{uA}_\infty \longrightarrow \mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty, \quad i = 1, 2,$$

are denoted by

$$\phi_i(\mu_n^S) = \mu_{n,i}^S.$$

In particular $\mu_{n,1}^\emptyset = \mu_{n,2}^\emptyset$, $n \geq 2$, and the underlying graded operad of $\mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty$ is freely generated by

$$\mu_{n,i}^S, \quad n > 0, \quad S \subset \{1, \dots, n\}, \quad i = 1, 2, \quad (n, S) \neq (1, \emptyset), \quad (i, S) \neq (2, \emptyset).$$

The differential is determined by the fact that ϕ_1 and ϕ_2 are morphisms of DG-operads.

We consider the exhausting filtration of $\mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty$,

$$\phi_1(\mathbf{uA}_\infty) = \mathbf{0}_0 \subset \mathbf{0}_1 \subset \mathbf{0}_2 \subset \dots \subset \mathbf{0}_{m-1} \subset \mathbf{0}_m \subset \dots \subset \mathbf{0}_\infty = \mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty,$$

such that $\mathbf{0}_m$ is freely generated as a graded operad by

$$\mu_{n,i}^{S_i}, \quad n > 0, \quad i = 1, 2, \quad S_i \subset \{1, \dots, n\}, \quad (n, S_i) \neq (1, \emptyset), \quad 0 < |S_i| \leq m.$$

In the following lemma, given $S = \{j_1, \dots, j_m\} \subset \{1, \dots, n\}$ and $l \in \mathbb{Z}$, we denote

$$S + l = \{j_1 + l, \dots, j_m + l\},$$

and we always assume that $j_1 < \dots < j_m$.

Lemma 4.1. *For any $m > 0$ there is a unique strong deformation retraction of DG-operads*

$$\mathbf{0}_{m-1} \xrightleftharpoons[r]{l} \mathbf{0}_m \xrightarrow{h}$$

such that l is the inclusion and if $|S| = m$ then

$$\begin{aligned} h(\mu_{n,2}^S) &= \sum_{p+q=n+2} (-1)^{q(p+\min S)+n} \mu_{p,1}^{\{\min S\}} \circ_{\min S} \mu_{q,2}^{S+1-\min S} \\ &\quad + \sum_{p+q=n+2} (-1)^{q(p+\min S-m+1)+n-m} \mu_{p,2}^{S+2-q} \circ_{\min S+1-q} \mu_{q,1}^{\{q\}}. \end{aligned}$$

Proof. For $i = 1$ or $|S| < m$ we set

$$h(\mu_{n,i}^S) = 0.$$

By Lemma 3.1 there is a unique morphism of DG-operads $f: \mathbf{0}_m \rightarrow \mathbf{0}_m$ and homotopy $h: f \Rightarrow 1_{\mathbf{0}_m}$ extending the definition of h on generators. Moreover, $hl = 0$ since h vanishes on the generators of $\mathbf{0}_{m-1}$. Therefore we only have to check that $f(\mu_{n,2}^S) \in \mathbf{0}_{m-1}$ for $|S| = m$.

We first consider the case $m = 1$. Then $S = \{j\} \subset \{1, \dots, n\}$, and for $k = 1, 2$,

$$\begin{aligned}
h(\mu_{n,2}^{\{j\}}) &= \sum_{p+q=n+2} (-1)^{q(p+j)+n} \mu_{p,1}^{\{j\}} \circ_j \mu_{q,2}^{\{1\}} \\
&\quad + \sum_{p+q=n+2} (-1)^{q(p+j)+n-1} \mu_{p,2}^{\{j+2-q\}} \circ_{j+1-q} \mu_{q,1}^{\{q\}}, \\
d(\mu_{n,k}^{\{j\}}) &= \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} (-1)^{qp+(q-1)i} \mu_{p,1}^{\emptyset} \circ_i \mu_{q,k}^{\{j-i+1\}} \\
&\quad + \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq j-q}} (-1)^{q(p-1)+(q-1)i} \mu_{p,k}^{\{j-q+1\}} \circ_i \mu_{q,1}^{\emptyset} \\
&\quad + \sum_{\substack{p+q=n+1 \\ j \leq i < p}} (-1)^{q(p-1)+(q-1)(i+1)} \mu_{p,k}^{\{j\}} \circ_i \mu_{q,1}^{\emptyset}, \quad \text{if } n \neq 2, \\
d(\mu_{2,k}^{\{j\}}) &= \mu_{2,1}^{\emptyset} \circ_j \mu_{1,k}^{\{1\}} - \text{id}.
\end{aligned}$$

On the one hand,

$$\begin{aligned}
dh(\mu_{n,2}^{\{j\}}) &= \\
\text{(a)} \quad &\sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s \\ s+t \neq 3}} (-1)^{q(s+t-1+j)+n+ts+(t-1)i} (\mu_{s,1}^{\emptyset} \circ_i \mu_{t,1}^{\{j-i+1\}}) \circ_j \mu_{q,2}^{\{1\}} \\
\text{(b)} \quad &+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j-t}} (-1)^{q(s+t-1+j)+n+t(s-1)+(t-1)i} (\mu_{s,1}^{\{j-t+1\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_j \mu_{q,2}^{\{1\}} \\
\text{(c)} \quad &+ \sum_{\substack{s+t+q=n+3 \\ j \leq i < s}} (-1)^{q(s+t-1+j)+n+t(s-1)+(t-1)(i+1)} (\mu_{s,1}^{\{j\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_j \mu_{q,2}^{\{1\}} \\
\text{(d)} \quad &\overbrace{(\mu_{2,1}^{\emptyset} \circ_1 \mu_{1,1}^{\{1\}}) \circ_1 \mu_{n,2}^{\{1\}} - \mu_{n,2}^{\{1\}}} \quad (\text{only if } j = 1) \\
\text{(e)} \quad &+ \sum_{\substack{p+s+t=n+3 \\ s+t \neq 3}} (-1)^{(s+t-1)(p+j)+n+p+1+ts+t-1} \mu_{p,1}^{\{j\}} \circ_j (\mu_{s,1}^{\emptyset} \circ_1 \mu_{t,2}^{\{1\}}) \\
\text{(f)} \quad &+ \sum_{\substack{p+s+t=n+3 \\ 1 \leq i < s}} (-1)^{(s+t-1)(p+j)+n+p+1+t(s-1)+(t-1)(i+1)} \mu_{p,1}^{\{j\}} \circ_j (\mu_{s,2}^{\{1\}} \circ_i \mu_{t,1}^{\emptyset}) \\
\text{(g)} \quad &\overbrace{-\mu_{n,1}^{\{j\}} \circ_j (\mu_{2,1}^{\emptyset} \circ_1 \mu_{1,2}^{\{1\}}) + \mu_{n,1}^{\{j\}}} \quad (\text{only if } j < n) \\
\text{(h)} \quad &+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s \\ s+t \neq 3}} (-1)^{q(s+t-1+j)+n-1+ts+(t-1)i} (\mu_{s,1}^{\emptyset} \circ_i \mu_{t,2}^{\{j+3-q-i\}}) \circ_{j+1-q} \mu_{q,1}^{\{q\}}
\end{aligned}$$

$$\begin{aligned}
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j+2-q-t}} (-1)^{q(s+t-1+j)+n-1+t(s-1)+(t-1)i} (\mu_{s,2}^{\{j+3-q-t\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_{j+1-q} \mu_{q,1}^{\{q\}}}^{(i)} \\
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ j+2-q \leq i < s}} (-1)^{q(s+t-1+j)+n-1+t(s-1)+(t-1)(i+1)} (\mu_{s,2}^{\{j+2-q\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_{j+1-q} \mu_{q,1}^{\{q\}}}^{(j)} \\
& - \overbrace{(\mu_{2,1}^{\emptyset} \circ_2 \mu_{1,2}^{\{1\}}) \circ_1 \mu_{n,1}^{\{n\}} + \mu_{n,1}^{\{n\}}}^{(k)} \quad (\text{only if } j = n) \\
& + \overbrace{\sum_{\substack{p+s+t=n+3 \\ s+t \neq 3}} (-1)^{(s+t-1)(p+j)+n-1+p+1+ts+(t-1)s} \mu_{p,2}^{\{j+3-s-t\}} \circ_{j+2-s-t} (\mu_{s,1}^{\emptyset} \circ_s \mu_{t,1}^{\{t\}})}^{(l)} \\
& + \overbrace{\sum_{\substack{p+s+t=n+3 \\ 1 \leq i < s}} (-1)^{(s+t-1)(p+j)+n-1+p+1+t(s-1)+(t-1)i} \mu_{p,2}^{\{j+3-s-t\}} \circ_{j+2-s-t} (\mu_{s,1}^{\{s\}} \circ_i \mu_{t,1}^{\emptyset})}^{(m)} \\
& + \overbrace{\mu_{n,2}^{\{j\}} \circ_{j-1} (\mu_{2,1}^{\emptyset} \circ_2 \mu_{1,1}^{\{1\}}) - \mu_{n,2}^{\{j\}}}^{(n)} \quad (\text{only if } j > 1).
\end{aligned}$$

After some reindexing, one can easily notice that the missing summand of (a), (e), (h), and (l) for $s+t=3$ is (d), (g), (k), and (n), respectively. Moreover, the summand $i=j$ of (c) is $-(e)-(g)$, and the summand $i=j+2-q-t$ of (i) is $-(l)-(n)$. Furthermore, using the associativity of composition in an operad we obtain that,

$$\begin{aligned}
dh(\mu_{n,2}^{\{j\}}) &= -\mu_{n,2}^{\{j\}} + \mu_{n,1}^{\{j\}} \\
\text{(a) + (d)} &+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s}} (-1)^{q(s+t-1+j)+n+ts+(t-1)i} (\mu_{s,1}^{\emptyset} \circ_i \mu_{t,1}^{\{j-i+1\}}) \circ_j \mu_{q,2}^{\{1\}} \\
\text{(b)} &+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j-t}} (-1)^{q(s+t-1+j)+n+t(s-1)+(t-1)i} (\mu_{s,1}^{\{j-t+1\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_j \mu_{q,2}^{\{1\}} \\
&+ \overbrace{\sum_{\substack{p+s+t=n+3 \\ j \leq i < p+s-1}} (-1)^{s(p+j)+t(p+s+i+1)+n+i+1} (\mu_{p,1}^{\{j\}} \circ_j \mu_{s,2}^{\{1\}}) \circ_i \mu_{t,1}^{\emptyset}}^{(c)+(e)+(f)+(g)}
\end{aligned}$$

$$\begin{aligned}
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s}} (-1)^{q(s+t-1+j)+n-1+ts+(t-1)i} (\mu_{s,1}^{\emptyset} \circ_i \mu_{t,2}^{\{j+3-q-i\}}) \circ_{j+1-q} \mu_{q,1}^{\{q\}}}^{(h)+(k)} \\
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ j+2-q \leq i < s}} (-1)^{q(s+t-1+j)+n-1+t(s-1)+(t-1)(i+1)} (\mu_{s,2}^{\{j+2-q\}} \circ_i \mu_{t,1}^{\emptyset}) \circ_{j+1-q} \mu_{q,1}^{\{q\}}}^{(j)} \\
& + \overbrace{\sum_{\substack{p+s+t=n+3 \\ 1 \leq i \leq j-t}} (-1)^{s(p+j+1)+t(p+i)+n+i+1} (\mu_{p,2}^{\{j+3-s-t\}} \circ_{j+2-s-t} \mu_{s,1}^{\{s\}}) \circ_i \mu_{t,1}^{\emptyset}}^{(i)+(l)+(m)+(n)}.
\end{aligned}$$

One the other hand, if $n \neq 2$,

$$\begin{aligned}
hd(\mu_{n,2}^{\{j\}}) = & \sum_{\substack{p+s+t=n+3 \\ 1 \leq i \leq p}} (-1)^{(s+t)p+(s+t-1)i+p+t(s+j-i+1)+s+t} \mu_{p,1}^{\emptyset} \circ_i (\mu_{s,1}^{\{j-i+1\}} \circ_{j-i+1} \mu_{t,2}^{\{1\}}) \\
& + \sum_{\substack{p+s+t=n+3 \\ 1 \leq i \leq p}} (-1)^{(s+t)p+(s+t-1)i+p+t(s+j-i+1)+s+t-1} \mu_{p,1}^{\emptyset} \circ_i (\mu_{s,2}^{\{j+3-i-t\}} \circ_{j+2-i-t} \mu_{t,1}^{\{t\}}) \\
& + \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j-q}} (-1)^{q(s+t-1)+(q-1)i+t(s+j-q+1)+s+t} (\mu_{s,1}^{\{j-q+1\}} \circ_{j-q+1} \mu_{t,2}^{\{1\}}) \circ_i \mu_{q,1}^{\emptyset} \\
& + \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j-q}} (-1)^{q(s+t-1)+(q-1)i+t(s+j-q+1)+s+t-1} (\mu_{s,2}^{\{j-q-t+3\}} \circ_{j-q-t+2} \mu_{t,1}^{\{t\}}) \circ_i \mu_{q,1}^{\emptyset} \\
& + \sum_{\substack{s+t+q=n+1 \\ j \leq i < s+t-1}} (-1)^{q(s+t-1)+(q-1)(i+1)+t(s+j)+s+t} (\mu_{s,1}^{\{j\}} \circ_j \mu_{t,2}^{\{1\}}) \circ_i \mu_{q,1}^{\emptyset} \\
& + \sum_{\substack{s+t+q=n+1 \\ j \leq i < s+t-1}} (-1)^{q(s+t-1)+(q-1)(i+1)+t(s+j)+s+t-1} (\mu_{s,2}^{\{j+2-t\}} \circ_{j+1-t} \mu_{t,1}^{\{t\}}) \circ_i \mu_{q,1}^{\emptyset}.
\end{aligned}$$

Reindexing appropriately, one can easily check that these summands coincide up to multiplication by -1 with (a) + (d), (h) + (k), (b), (i) + (l) + (m) + (n), (c) + (e) + (f) + (g), and (j), respectively, therefore, for $n \neq 2$,

$$f(\mu_{n,2}^{\{j\}}) = \mu_{n,2}^{\{j\}} + dh(\mu_{n,2}^{\{j\}}) + hd(\mu_{n,2}^{\{j\}}) = \mu_{n,1}^{\{j\}} \in \mathbb{O}_0.$$

This equation also holds for $n = 2$ since in this case,

$$\begin{aligned}
hd(\mu_{2,2}^{\{1\}}) &= \mu_{2,1}^{\emptyset} \circ_1 (\mu_{2,1}^{\{1\}} \circ_1 \mu_{1,2}^{\{1\}} - \mu_{2,2}^{\{2\}} \circ_1 \mu_{1,1}^{\{1\}}), \\
hd(\mu_{2,2}^{\{2\}}) &= \mu_{2,1}^{\emptyset} \circ_2 (\mu_{2,1}^{\{1\}} \circ_1 \mu_{1,2}^{\{1\}} - \mu_{2,2}^{\{2\}} \circ_1 \mu_{1,1}^{\{1\}}), \\
dh(\mu_{2,2}^{\{1\}}) &= (-\mu_{3,1}^{\emptyset} \circ_1 \mu_{1,1}^{\{1\}} - \mu_{2,1}^{\emptyset} \circ_1 \mu_{2,1}^{\{1\}} + \mu_{2,1}^{\{1\}} \circ_1 \mu_{2,1}^{\emptyset}) \circ_1 \mu_{1,2}^{\{1\}} \\
&\quad + (\mu_{2,1}^{\emptyset} \circ_1 \mu_{1,1}^{\{1\}} - \text{id}) \circ_1 \mu_{2,2}^{\{1\}} - \mu_{2,1}^{\{1\}} \circ_1 (\mu_{2,1}^{\emptyset} \circ_1 \mu_{1,2}^{\{1\}} - \text{id}) \\
&\quad - (-\mu_{3,1}^{\emptyset} \circ_2 \mu_{1,2}^{\{1\}} - \mu_{2,1}^{\emptyset} \circ_1 \mu_{2,2}^{\{2\}} + \mu_{2,1}^{\emptyset} \circ_2 \mu_{2,2}^{\{1\}}) \circ_1 \mu_{1,1}^{\{1\}}, \\
dh(\mu_{2,2}^{\{2\}}) &= -(-\mu_{3,1}^{\emptyset} \circ_2 \mu_{1,1}^{\{1\}} - \mu_{2,1}^{\emptyset} \circ_1 \mu_{2,1}^{\{2\}} + \mu_{2,1}^{\emptyset} \circ_2 \mu_{2,1}^{\{1\}}) \circ_2 \mu_{1,2}^{\{1\}} \\
&\quad + (-\mu_{3,1}^{\emptyset} \circ_3 \mu_{1,2}^{\{1\}} + \mu_{2,1}^{\emptyset} \circ_2 \mu_{2,2}^{\{2\}} - \mu_{2,2}^{\{2\}} \circ_1 \mu_{2,1}^{\emptyset}) \circ_2 \mu_{1,1}^{\{1\}} \\
&\quad - (\mu_{2,1}^{\emptyset} \circ_2 \mu_{1,2}^{\{1\}} - \text{id}) \circ_1 \mu_{2,1}^{\{2\}} + \mu_{2,2}^{\{2\}} \circ_1 (\mu_{2,1}^{\emptyset} \circ_2 \mu_{1,1}^{\{1\}} - \text{id}).
\end{aligned}$$

This concludes the proof of the case $m = 1$.

For $m > 1$, $S = \{j_1, \dots, j_m\}$, we denote

$$S_r = \{j_r, \dots, j_m\}, \quad S'_r = S \setminus S_r, \quad 1 \leq r \leq m.$$

For $k = 1, 2$,

$$\begin{aligned}
d(\mu_{n,k}^S) &= \sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} (-1)^{qp+(q-1)i} \mu_{p,1}^{\emptyset} \circ_i \mu_{q,k}^{S+1-i} \\
&\quad + \sum_{\substack{p+q=n+1 \\ 1 \leq r \leq m+1 \\ j_{r-1} < j_r+1-q \\ j_{r-1} < i+r-1 < j_r}} (-1)^{q(p-m)+(q-1)(i+r-1)} \mu_{p,k}^{S'_r \cup (S_r+1-q)} \circ_i \mu_{q,1}^{\emptyset} \\
&\quad + \text{terms in } \mathcal{O}_{m-1}.
\end{aligned}$$

We follow the conventions $j_0 = 0$, $j_{m+1} = n+1$, and $S_{m+1} = \emptyset$.

Now, one can argue in the same way as in case $m = 1$. The case $m > 1$ is actually simpler. The formula for $dh(\mu_{n,2}^S)$ decomposes in several summands. We give a name from (a) to (k) to some summations in it. The remaining terms are $-\mu_{n,2}^S$ and some irrelevant terms in \mathcal{O}_{m-1} . The missing summand of (a) (resp. (i)) for $s+t=3$ is (d) (resp. (k)). Moreover, the summand $i = j_1$ of (c) is $-(e)$, and the summand $(r, i) = (1, j_1 + 2 - q - t)$ of (h) is $-(i) - (k)$. Furthermore, the formula for $hd(\mu_{n,2}^S)$ decomposes as the sum of six summations which cancel with (a), ..., (k) as indicated below. In order to check these facts one uses the associativity of composition in an operad and some reindexing.

$$dh(\mu_{n,2}^S) = \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s \\ s+t \neq 3}} (-1)^{q(s+t-1+j_1)+n+ts+(t-1)i} (\mu_{s,1}^\emptyset \circ_i \mu_{t,1}^{\{j_1-i+1\}}) \circ_{j_1} \mu_{q,2}^{S+1-j_1} \quad (a)$$

$$+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq j_1-t}} (-1)^{q(s+t-1+j_1)+n+t(s-1)+(t-1)i} (\mu_{s,1}^{\{j_1-t+1\}} \circ_i \mu_{t,1}^\emptyset) \circ_{j_1} \mu_{q,2}^{S+1-j_1} \quad (b)$$

$$+ \sum_{\substack{s+t+q=n+3 \\ j_1 \leq i < s}} (-1)^{q(s+t-1+j_1)+n+t(s-1)+(t-1)(i+1)} (\mu_{s,1}^{\{j_1\}} \circ_i \mu_{t,1}^\emptyset) \circ_{j_1} \mu_{q,2}^{S+1-j_1} \quad (c)$$

$$+ \overbrace{(\mu_{2,1}^\emptyset \circ_1 \mu_{1,1}^{\{1\}}) \circ_1 \mu_{n,2}^S - \mu_{n,2}^S}^{(d)} \quad (\text{only if } j_1 = 1) \\ + \sum_{p+s+t=n+3} (-1)^{(s+t-1)(p+j_1)+n+p+1+ts+t-1} \mu_{p,1}^{\{j_1\}} \circ_{j_1} (\mu_{s,1}^\emptyset \circ_1 \mu_{t,2}^{S+1-j_1}) \quad (e)$$

$$+ \sum_{\substack{p+s+t=n+3 \\ 1 \leq r \leq m+1 \\ j_{r-1} < j_r+1-t \\ j_{r-1} < i+r-2+j_1 < j_r}} (-1)^{(s+t-1)(p+j_1)+n+p+1+t(s-m)+(t-1)(i+r-1)} \mu_{p,1}^{\{j_1\}} \circ_{j_1} (\mu_{s,2}^{(S'_r+1-j_1) \cup (S_r+2-j_1-t)} \circ_i \mu_{t,1}^\emptyset) \quad (f)$$

$$+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq i \leq s}} (-1)^{q(s+t+j_1-m)+n-m+ts+(t-1)i} (\mu_{s,1}^\emptyset \circ_i \mu_{t,2}^{S+3-i-q}) \circ_{j_1+1-q} \mu_{q,1}^{\{q\}} \quad (g)$$

$$+ \sum_{\substack{s+t+q=n+3 \\ 1 \leq r \leq m+1 \\ j_{r-1} < j_r+1-t \\ j_{r-1} < i+r+q-3 < j_r}} (-1)^{q(s+t+j_1-m)+n-m+t(s-m)+(t-1)(i+r-1)} (\mu_{s,2}^{(S'_r+2-q) \cup (S_r+3-q-t)} \circ_i \mu_{t,1}^\emptyset) \circ_{j_1+1-q} \mu_{q,1}^{\{q\}} \quad (h)$$

$$+ \sum_{\substack{p+s+t=n+3 \\ s+t \neq 3}} (-1)^{(s+t-1)(p+j_1-m+1)+n-m+p+m+ts+(t-1)s} \mu_{p,2}^{S+3-s-t} \circ_{j_1+2-s-t} (\mu_{s,1}^\emptyset \circ_s \mu_{t,1}^{\{t\}}) \quad (i)$$

$$\begin{aligned}
& + \sum_{\substack{p+s+t=n+3 \\ 1 \leq i < s}} (-1)^{(s+t-1)(p+j_1-m+1)+n-m+p+m+t(s-1)+(t-1)i} \mu_{p,2}^{S+3-s-t} \circ_{j_1+2-s-t} (\mu_{s,1}^{\{s\}} \circ_i \mu_{t,1}^{\emptyset}) \quad (j) \\
& + \underbrace{\mu_{n,2}^S \circ_{j_1-1} (\mu_{2,1}^{\emptyset} \circ_2 \mu_{1,1}^{\{1\}})}_{(k)} - \mu_{n,2}^S \quad (\text{only if } j_1 > 1) \\
& + \text{terms in } \mathcal{O}_{m-1}, \\
hd(\mu_{n,2}^S) = & \sum_{\substack{p+s+t=n+3 \\ 1 \leq i \leq p}} (-1)^{(s+t)p+(s+t-1)i+p+t(s+j_1+i-1)+s+t} \mu_{p,1}^{\emptyset} \circ_i (\mu_{s,1}^{\{j_1-i+1\}} \circ_{j_1-i+1} \mu_{t,2}^{S+1-j_1}) \quad - (a) - (d) \\
& + \sum_{\substack{p+s+t=n+3 \\ 1 \leq i \leq p}} (-1)^{(s+t)p+(s+t-1)i+p+t(s+j_1+i-m)+s+t-m} \mu_{p,1}^{\emptyset} \circ_i (\mu_{s,2}^{S+3-i-t} \circ_{j_1+2-i-t} \mu_{t,1}^{\{t\}}) \quad - (g) \\
& + \sum_{\substack{s+t+q=n+3 \\ 1 < r \leq m+1 \\ j_{r-1} < j_r+1-q \\ j_{r-1} < i+r-1 < j_r}} (-1)^{q(s+t-m)+(q-1)(i+r-1)+t(s+j_1)+s+t} (\mu_{s,1}^{\{j_1\}} \circ_{j_1} \mu_{t,2}^{S'_r+(1-j_1) \cup (S_r+2-q-j_1)}) \circ_i \mu_{q,1}^{\emptyset} \quad - (c) - (e) - (f) \\
& + \sum_{\substack{s+t+q=n+3 \\ 1 \leq i < j_1-q}} (-1)^{q(s+t-m)+(q-1)i+t(s+j_1+1-q)+s+t} (\mu_{s,1}^{\{j_1+1-q\}} \circ_{j_1+1-q} \mu_{t,2}^{S+1-j_1}) \circ_i \mu_{q,1}^{\emptyset} \quad - (b) \\
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ 1 < r \leq m+1 \\ j_{r-1} < j_r+1-q \\ j_{r-1} < i+r-1 < j_r}} (-1)^{q(s+t-m)+(q-1)(i+r-1)+t(s+j_1-m+1)+s+t-m} (\mu_{s,2}^{(S'_r+2-t) \cup (S_r+3-q-t)} \circ_{j_1+1-t} \mu_{t,1}^{\{t\}}) \circ_i \mu_{q,1}^{\emptyset}}_{-(\text{summands of (h) for } r > 1)} \\
& + \overbrace{\sum_{\substack{s+t+q=n+3 \\ 1 < r \leq m+1 \\ j_{r-1} < j_r+1-q \\ j_{r-1} < i+r-1 < j_r}} (-1)^{q(s+t-m)+(q-1)i+t(s+j_1-q-m)+s+t-m} (\mu_{s,2}^{S+3-q-t} \circ_{j_1+2-q-t} \mu_{t,1}^{\{t\}}) \circ_i \mu_{q,1}^{\emptyset}}_{-(\text{summands of (h) for } r=1) - (i) - (j)} \\
& + \sum_{\substack{s+t+q=n+3 \\ 1 \leq i < j_1+1-q}} (-1)^{q(s+t-m)+(q-1)i+t(s+j_1-q-m)+s+t-m} (\mu_{s,2}^{S+3-q-t} \circ_{j_1+2-q-t} \mu_{t,1}^{\{t\}}) \circ_i \mu_{q,1}^{\emptyset}.
\end{aligned}$$

We finally conclude that

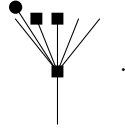
$$f(\mu_{n,2}^S) = \mu_{n,2}^S + dh(\mu_{n,2}^S) + hd(\mu_{n,2}^S) \in \mathcal{O}_{m-1},$$

hence we are done. \square

Remark 4.2. We here give a certain interpretation of the formula for h in Lemma 4.1 in terms of trees. In order to distinguish generators $\mu_{n,1}^S$ and $\mu_{n,2}^S$, we draw its inner vertices in round or square shape, respectively,

$$\begin{aligned} \mu_{5,1}^{\{2,3\}} &\sim \begin{array}{c} \bullet \bullet \\ \diagup \diagdown \\ \bullet \\ | \end{array}, & \mu_{1,1}^{\{1\}} &\sim \begin{array}{c} \circ \\ | \end{array}, \\ \mu_{5,2}^{\{2,3\}} &\sim \begin{array}{c} \blacksquare \blacksquare \\ \diagup \diagdown \\ \blacksquare \\ | \end{array}, & \mu_{1,2}^{\{1\}} &\sim \begin{array}{c} \square \\ | \end{array}, \\ \mu_{n,1}^\emptyset &\sim \begin{array}{c} \overbrace{}^n \\ \diagup \diagdown \\ \bullet \\ | \end{array} \sim \begin{array}{c} \overbrace{}^n \\ \diagup \diagdown \\ \blacksquare \\ | \end{array} \sim \mu_{n,2}^\emptyset. \end{aligned}$$

Let $n \neq 1$. Consider the tree obtained from $\mu_{n,2}^S$ by adjoining a new edge with a round cork right before the first edge of $\mu_{n,2}^S$ with a (square) cork,



This tree does not represent any element in the operad $\mathbf{uA}_\infty \cup_{\mathbf{A}_\infty} \mathbf{uA}_\infty$, but some summands in its hypothetical boundary (i.e. image by the differential d) do make sense,

$$\begin{aligned} \mu_{6,2}^{\{3,4\}} \circ_2 \mu_{1,1}^{\{1\}} &\sim \begin{array}{c} \circ \blacksquare \blacksquare \\ \diagup \diagdown \\ \blacksquare \\ | \end{array}, & \mu_{5,2}^{\{2,3\}} \circ_1 \mu_{2,1}^{\{2\}} &\sim \begin{array}{c} \bullet \bullet \blacksquare \blacksquare \\ \diagup \diagdown \\ \bullet \\ | \end{array}, \\ \mu_{4,1}^{\{2\}} \circ_2 \mu_{3,2}^{\{1,2\}} &\sim \begin{array}{c} \blacksquare \blacksquare \\ \diagup \diagdown \\ \bullet \bullet \\ | \end{array}, & &\dots \end{aligned}$$

The formula for $h(\mu_{n,2}^S)$ consists of these terms of the hypothetical boundary, up to sign $(-1)^{n+\min S-|S|+1}$.

For $n = 1$,

$$h \left(\begin{array}{c} \square \\ | \end{array} \right) = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \square \\ | \end{array} - \begin{array}{c} \circ \quad \blacksquare \\ \diagdown \quad \diagup \\ \bullet \quad \blacksquare \\ | \end{array}.$$

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